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Multipartite Generalized Bell Inequality with an Arbitrary Number of Settings

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Abstract We derive a generalized Bell inequality for N qubits which involves an arbitrary number of settings for each of the local measuring apparatuses. The inequality forms a necessary condition for the existence of a local realistic model for the values of a correlation function, given in a n-setting Bell experiment. We show that a local realistic model for the values of a correlation function, given in a two-setting Bell experiment, cannot construct a local realistic model for the values of a correlation function, given in an arbitrary number of n-setting Bell experiment, even though there exist two-setting models for the n measurement directions chosen in the given n-setting experiment. Therefore, the property of two-setting model is different from the property of n-setting model. We discuss classification of local realistic theories in further detail more than the result presented in (J. Phys. A: Math. Theor. 41:155308, 2008). The generalized Bell inequality covers the previous results correctly.

Keywords Local realism · Bell inequalities

1 Introduction

There is much research about local realism [1–4]. The locality condition is the assumption of a finite speed of influences (i.e., it is a consequence of the special relativity theory). Some quantum predictions violate Bell inequalities [2], which are conditions that a local realistic theory must satisfy. Hence, the quantum theory does not accept a local realistic interpretation.

Experimental efforts (Bell experiments) of a violation of local realism can be seen in Refs. [5–13]. Other type of inequalities are given in Refs. [14, 15]. Bell inequalities with settings other than spin polarizations can be seen in Ref. [16]. It would be very interesting to consider situations in which symmetries of physical situations additionally constrain local realistic theories.

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Therefore, we consider symmetrical n-setting Bell experiment for a system described by multipartite states in the case where n dichotomic observables are measured per site. If n is two, we consider two-setting Bell experiment. If n is three, we consider three-setting Bell experiment and so on.

It is shown [17, 18] that an explicit local realistic model for the values of a correlation function, given in a two-setting Bell experiment (two-setting model), works only for the specific set of settings in the given experiment, but cannot construct a local realistic model for the values of a correlation function, given in a Bell experiment with continuous-infinite settings lying in a plane (plane-infinite-setting model), even though there exist two-setting models for all directions in the plane. Therefore, the property of two-setting model is different from the property of plane-infinite-setting model.

Further, in specific type of quantum states, it is shown [19, 20] that the generalized Bell inequality under the assumption of the existence of a local realistic model which is rotationally invariant (sphere-infinite-setting model) disproves two-setting model stronger than the generalized Bell inequality under the assumption of the existence of a local realistic model which is rotationally invariant with respect to a plane (i.e., plane-infinite-setting model). Therefore, the property of two-setting model is different from the property of sphere-infinite-setting model. Also we see that the property of plane-infinite-setting model is different from the property of sphere-infinite-setting model. More recently, it is shown [21] that the property of two-setting model is different from the property of three-setting model.

We thus see that there is a division among the measurement settings, those that admit local realistic models which are rotationally invariant (sphere-infinite-setting model), those that admit local realistic models which are rotationally invariant with respect to a plane (plane-infinite-setting model), and those that do not (e.g., two-setting model and three-setting model). This is another manifestation of the underlying contextual nature of local realistic theories of quantum experiments.

In this paper, we consider the difference between two-setting model and general number of n-setting model. This discussion is generalization of the result presented in Refs. [17, 21]. We derive a generalized Bell inequality with an arbitrary number of settings. We show that two-setting model cannot construct a local realistic model for the values of a correlation function, given in a n-setting Bell experiment (n-setting model), even though there exist two-setting models for the n measurement directions chosen in the given n-setting experiment. The property of two-setting model is different from the property of n-setting model. To this end, we derive a generalized Bell inequality. The inequality forms a necessary condition for the existence of n-setting model. Although the inequality involves n-setting, it can be experimentally tested by using two orthogonal local measurement settings. This is a direct consequence of the assumed form of rotationally invariant correlation like (2). We see that our generalized Bell inequality under the assumption of the existence of n-setting model disproves two-setting model for the actually measured values of the correlation function. Therefore, the property of two-setting model is different from the property of n-setting model. Further, the generalized Bell inequality covers the previous results in Refs. [17, 21] correctly.

Our result provides classification of local realistic theories in further detail (more than the result presented in Ref. [21]). In order to say that some model is different from another model, we need criterion. Our criterion is as follows [21]. *We may say that model (A1) is different from model (A2) if model (A1) does not have the property which model (A2) has.* We shall stand to this approach.

Then, we can see that there are many types of local realistic theories along with measurement settings. First, there is two-setting model. It is explicitly constructed by standard two-setting Bell inequalities [22]. However, this model is disproved by several generalized Bell inequalities. The patterns of the disqualification are different form each other along with measurement settings. Therefore, one furthermore has more different types of models. These are n-setting model ($3 \le n$), plane-infinite-model, and sphere-infinite-model, as we mentioned above.

2 Multipartite Generalized Bell Inequality with an Arbitrary Number of Settings

Assume that we have a set of N spins $\frac{1}{2}$. Each of them is in a separate laboratory. As is well known the measurements (observables) for such spins are parameterized by a unit vector $\vec{n}_j$, $j = 1, 2, \ldots, N$ (its direction along which the spin component is measured). The results of measurements are ± 1 (in $\hbar/2$ unit). One can introduce the "Bell" correlation function, which is the average of the product of the local results:

$$E(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \langle r_1(\vec{n}_1) r_2(\vec{n}_2) \cdots r_N(\vec{n}_N) \rangle_{avg}, \tag{1}$$

where $r_j(\vec{n}_j)$ is the local result, ± 1, which is obtained if the measurement direction is set at $\vec{n}_j$. If experimental correlation function admits a rotationally invariant tensor structure familiar from quantum mechanics, we can introduce the following form:

$$E(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \hat{T} \cdot (\vec{n}_1 \otimes \vec{n}_2 \otimes \cdots \otimes \vec{n}_N), \tag{2}$$

where $\otimes$ denotes the tensor product, $\cdot$ the scalar product in R^{3N} and $\hat{T}$ is the correlation tensor given by

$$T_{i_1 \ldots i_N} \equiv E(\vec{x}_1^{(i_1)}, \vec{x}_2^{(i_2)}, \ldots, \vec{x}_N^{(i_N)}), \tag{3}$$

where $\vec{x}_j^{(i_j)}$ is a unit directional vector of the local coordinate system of the jth observer; $i_j = 1, 2, 3$ gives the full set of orthogonal vectors defining the local Cartesian coordinates. That is, the components of the correlation tensor are experimentally accessible by measuring the correlation function at the directions given by the basis vectors of local coordinate systems. Obviously the assumed form of (2) implies rotational invariance, because the correlation function is a scalar. Rotational invariance simply states that the value of $E(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N)$ cannot depend on the local coordinate systems used by the N observers. Assume that one knows the values of all 3^N components of the correlation tensor, $T_{i_1 \ldots i_N}$, which are obtainable by performing specific 3^N measurements of the correlation function, compare (3). Then, with the use of formula (2) one can reproduce the value of the correlation functions for all other possible sets of local settings.

Using this rotationally invariant structure of the correlation function, we shall derive a necessary condition for the existence of a n-setting local realistic model for the values of the experimental correlation function (2) given in a n-setting Bell experiment.

If the correlation function is described by a n-setting local realistic theory, then the correlation function must be simulated by the following structure

$$E_{LR}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \int d\lambda \rho(\lambda) I^{(1)}(\vec{n}_1, \lambda) I^{(2)}(\vec{n}_2, \lambda) \cdots I^{(N)}(\vec{n}_N, \lambda), \tag{4}$$

where λ is some local hidden variable, $\rho(\lambda)$ is a probabilistic distribution, and $I^{(j)}(\vec{n}_j, \lambda)$ is the predetermined "hidden" result of the measurement of the n dichotomic observables $\vec{n} \cdot \vec{\sigma}$ with values ± 1.

Let us parametrize the n unit vectors in the plane defined by $\vec{x}_j^{(1)}$ and $\vec{x}_j^{(2)}$ in the following way

$$\vec{n}_j(\alpha_j^{l_j}) = \cos\alpha_j^{l_j}\vec{x}_j^{(1)} + \sin\alpha_j^{l_j}\vec{x}_j^{(2)}, \quad j = 1, 2, \ldots, N. \tag{5}$$

The phases $\alpha_j^{l_j}$ that experimentalists are allowed to set are chosen as

$$\alpha_j^{l_j} = (l_j - 1)\pi/n, \quad l_j = 1, 2, \ldots, n. \tag{6}$$

We shall show that the scalar product of "n-setting" local realistic correlation function

$$E_{LR}(\alpha_1^{l_1}, \alpha_2^{l_2}, \ldots, \alpha_N^{l_N}) = \int d\lambda\rho(\lambda)I^{(1)}(\alpha_1^{l_1}, \lambda)I^{(2)}(\alpha_2^{l_2}, \lambda)\cdots I^{(N)}(\alpha_N^{l_N}, \lambda), \tag{7}$$

with rotationally invariant correlation function, that is

$$E(\alpha_1^{l_1}, \alpha_2^{l_2}, \ldots, \alpha_N^{l_N}) = \hat{T}\cdot\vec{n}_1(\alpha_1^{l_1})\otimes\vec{n}_2(\alpha_2^{l_2})\otimes\cdots\otimes\vec{n}_N(\alpha_N^{l_N}), \tag{8}$$

is bounded by a specific number dependent on $\hat{T}$. We define the scalar product (E_{LR}, E) as follows. We see that the maximal possible value of (E_{LR}, E) is bounded as:

$$\begin{aligned}(E_{LR}, E) &= \sum_{l_1=1}^{n}\sum_{l_2=1}^{n}\cdots\sum_{l_N=1}^{n} E_{LR}(\alpha_1^{l_1}, \alpha_2^{l_2}, \ldots, \alpha_N^{l_N})E(\alpha_1^{l_1}, \alpha_2^{l_2}, \ldots, \alpha_N^{l_N}) \\ &\le \left(\frac{1}{\sin(\frac{\pi}{2n})}\right)^N T_{max},\end{aligned} \tag{9}$$

where T_{max} is the maximal possible value of the correlation tensor component, i.e.,

$$T_{max} \equiv \max_{\beta_1,\beta_2,\ldots,\beta_N} E(\beta_1, \beta_2, \ldots, \beta_N), \tag{10}$$

where β_j is some angle.

A necessary condition for the existence of "n-setting" local realistic description E_{LR} of the experimental correlation function

$$E(\alpha_1^{l_1}, \alpha_2^{l_2}, \ldots, \alpha_N^{l_N}) = E(\vec{n}_1(\alpha_1^{l_1}), \ldots, \vec{n}_N(\alpha_N^{l_N})), \tag{11}$$

that is for E_{LR} to be equal to E for the n measurement directions, is that one has $(E_{LR}, E) = (E, E)$. This implies the possibility of modeling E by "n-setting" local realistic correlation function E_{LR} given in (7) with respect to the n measurement directions. If we have $(E_{LR}, E) < (E, E)$, then the experimental correlation function cannot be explainable by n-setting local realistic model. (Note that, due to the summation in (9), we are looking for n-setting model.)

In what follows, we derive the upper bound (9). Since the local realistic model is an average over λ, it is enough to find the bound of the following expression

$$\sum_{l_1=1}^{n}\cdots\sum_{l_N=1}^{n} I^{(1)}(\alpha_1^{l_1}, \lambda)\cdots I^{(N)}(\alpha_N^{l_N}, \lambda)\sum_{i_1,i_2,\ldots,i_N=1,2} T_{i_1i_2\ldots i_N}c_1^{i_1}c_2^{i_2}\cdots c_N^{i_N}, \tag{12}$$

where

$$(c_j^1, c_j^2) = (\cos\alpha_j^{l_j}, \sin\alpha_j^{l_j}), \tag{13}$$

and

$$T_{i_1 i_2 \ldots i_N} = \hat{T} \cdot (\vec{x}_1^{(i_1)} \otimes \vec{x}_2^{(i_2)} \otimes \cdots \otimes \vec{x}_N^{(i_N)}), \tag{14}$$

compare (2) and (3).

Let us analyze the structure of this sum (12). Notice that (12) is a sum, with coefficients given by $T_{i_1 i_2 \ldots i_N}$, which is a product of the following sums:

$$\sum_{l_j=1}^{n} I^{(j)}(\alpha_j^{l_j}, \lambda) \cos\alpha_j^{l_j}, \tag{15}$$

and

$$\sum_{l_j=1}^{n} I^{(j)}(\alpha_j^{l_j}, \lambda) \sin\alpha_j^{l_j}. \tag{16}$$

We deal here with sums, or rather scalar products of $I^{(j)}(\alpha_j^{l_j}, \lambda)$ with two orthogonal vectors. One has

$$\sum_{l_j=1}^{n} \cos\alpha_j^{l_j} \sin\alpha_j^{l_j} = 0, \tag{17}$$

because,

$$2 \times \sum_{l_j=1}^{n} \cos\alpha_j^{l_j} \sin\alpha_j^{l_j} = \sum_{l_j=1}^{n} \sin 2\alpha_j^{l_j} = \mathrm{Im}\left(\sum_{l_j=1}^{n} e^{i2\alpha_j^{l_j}}\right). \tag{18}$$

Since $\sum_{l_j=1}^{n} e^{i(l_j-1)(2/n)\pi} = 0$, the last term vanishes.

Please note

$$\sum_{l_j=1}^{n} (\cos\alpha_j^{l_j})^2 = \sum_{l_j=1}^{n} \frac{1+\cos 2\alpha_j^{l_j}}{2} = n/2 \tag{19}$$

and

$$\sum_{l_j=1}^{n} (\sin\alpha_j^{l_j})^2 = \sum_{l_j=1}^{n} \frac{1-\cos 2\alpha_j^{l_j}}{2} = n/2, \tag{20}$$

because,

$$\sum_{l_j=1}^{n} \cos 2\alpha_j^{l_j} = \mathrm{Re}\left(\sum_{l_j=1}^{n} e^{i2\alpha_j^{l_j}}\right). \tag{21}$$

Since $\sum_{l_j=1}^{n} e^{i(l_j-1)(2/n)\pi} = 0$, the last term vanishes.

The normalized vectors

$$M_1 \equiv \sqrt{\frac{2}{n}}\left(\cos 0, \cos\frac{\pi}{n}, \ldots, \cos\frac{(n-1)\pi}{n}\right)$$

and

$$M_2 \equiv \sqrt{\frac{2}{n}}\left(\sin 0, \sin\frac{\pi}{n}, \ldots, \sin\frac{(n-1)\pi}{n}\right)$$

form a basis of a real two-dimensional plane, which we shall call $S^{(2)}$. Note further that any vector in $S^{(2)}$ is of the form

$$A \cdot M_1 + B \cdot M_2, \tag{22}$$

where A and B are constants, and that any normalized vector in $S^{(2)}$ is given by

$$\cos\psi M_1 + \sin\psi M_2 = \sqrt{\frac{2}{n}}\left(\cos(0-\psi), \cos\left(\frac{\pi}{n}-\psi\right), \ldots, \cos\left(\frac{(n-1)\pi}{n}-\psi\right)\right). \tag{23}$$

The norm $\|I^{(j)\|}\|$ of the projection of $I^{(j)}$ into the plane $S^{(2)}$ is given by the maximal possible value of the scalar product $I^{(j)}$ with any normalized vector belonging to $S^{(2)}$, that is

$$\begin{aligned}\|I^{(j)\|}\| &= \max_{\psi}\sum_{l_j=1}^{n} I^{(j)}(\alpha_j^{l_j}, \lambda)\sqrt{\frac{2}{n}}\cos(\alpha_j^{l_j}-\psi)\\ &= \sqrt{\frac{2}{n}}\max_{\psi}\operatorname{Re}(z\exp(i(-\psi))),\end{aligned} \tag{24}$$

where $z = \sum_{l_j=1}^{n} I^{(j)}(\alpha_j^{l_j}, \lambda)\exp(i\alpha_j^{l_j})$. We may assume $|I^{(j)}(\alpha_j^{l_j}, \lambda)| = 1$. We maximize the length of z (i.e., $|z|$) on the complex plane. The length of the sum of any two complex numbers $|z_1+z_2|^2$ is given by the law of cosines as $|z_1|^2+|z_2|^2+2|z_1||z_2|\cos\varphi$, where φ is the angle between the corresponding vectors. To maximize the length of the sum one should choose the summands as close as possible to each other. Since in our case all vectors being summed are rotated by multiples of $\frac{\pi}{n}$ from each other, the simplest optimal choice is to put all $I^{(j)}(\alpha_j^{l_j}, \lambda) = 1$. In this case one has:

$$|z| = \left|\sum_{i_j=1}^{n}\exp\left(i\frac{(i_j-1)\pi}{n}\right)\right| = \left|\frac{2}{1-\exp(i\frac{\pi}{n})}\right|, \tag{25}$$

where the last equality follows from the finite sum of numbers in the geometric progression (any term in the sum is given by the preceding term multiplied by $e^{i\pi/n}$). The denominator inside the modulus can be transformed to $\exp(i\frac{\pi}{2n})[\exp(-i\frac{\pi}{2n}) - \exp(i\frac{\pi}{2n})]$, which reduces to $-2i\exp(i\frac{\pi}{2n})\sin(\frac{\pi}{2n})$. Finally, the maximal length reads:

$$z = \frac{1}{\sin\left(\frac{\pi}{2n}\right)}, \tag{26}$$

where the modulus is no longer needed since the argument of sine is small.

Note that the minimum possible overall complex phase (modulo 2π) of $(z\exp(i(-\psi)))$ is 0. Then we obtain

$$\|I^{(j)||}\| \leq \sqrt{\frac{2}{n}}\frac{1}{\sin(\frac{\pi}{2n})}\cos 0 = \sqrt{\frac{2}{n}}\frac{1}{\sin(\frac{\pi}{2n})}.$$

That is, one has

$$\|I^{(j)||}\| \leq \sqrt{\frac{2}{n}}\frac{1}{\sin(\frac{\pi}{2n})}.$$

Since M_1 and M_2 are two orthogonal basis vectors in $S^{(2)}$, one has

$$\sum_{l_j=1}^{n} I^{(j)}(\alpha_j^{l_j},\lambda)\cdot\sqrt{\frac{2}{n}}\cos\alpha_j^{l_j} = \cos\beta_j\|I^{(j)||}\|, \tag{27}$$

and

$$\sum_{l_j=1}^{n} I^{(j)}(\alpha_j^{l_j},\lambda)\cdot\sqrt{\frac{2}{n}}\sin\alpha_j^{l_j} = \sin\beta_j\|I^{(j)||}\|, \tag{28}$$

where β_j is some angle. Using this fact one can put the value of (12) into the following form

$$\left(\sqrt{\frac{n}{2}}\right)^N \prod_{j=1}^{N}\|I^{(j)||}\| \sum_{i_1,i_2,\dots,i_N=1,2} T_{i_1 i_2\dots i_N} d_1^{i_1} d_2^{i_2}\cdots d_N^{i_N}, \tag{29}$$

where

$$(d_j^1, d_j^2) = (\cos\beta_j, \sin\beta_j). \tag{30}$$

Let us look at the expression

$$\sum_{i_1 i_2\dots i_N=1,2} T_{i_1 i_2\dots i_N} d_1^{i_1} d_2^{i_2}\cdots d_N^{i_N}. \tag{31}$$

Formula (30) shows that we deal here with two dimensional unit vectors $\vec{d}_j = (d_j^1, d_j^2)$, $j = 1, 2, \dots, N$, that is (31) is equal to $\hat{T}\cdot(\vec{d}_1\otimes\vec{d}_2\otimes\cdots\otimes\vec{d}_N)$, i.e., it is a component of the tensor $\hat{T}$ in the directions specified by the vectors $\vec{d}_j$. If one knows all the values of $T_{i_1 i_2\dots i_N}$, one can always find the maximal possible value of such a component, and it is equal to T_{max}, of (10).

Therefore since

$$\|I^{(j)||}\| \leq \sqrt{\frac{2}{n}}\frac{1}{\sin(\frac{\pi}{2n})}$$

the maximal value of (29) is

$$\left(\frac{1}{\sin(\frac{\pi}{2n})}\right)^N T_{max},$$

and finally one has

$$(E_{LR}, E) \leq \left(\frac{1}{\sin(\frac{\pi}{2n})}\right)^N T_{max}. \tag{32}$$

Please note that the relation (32) is a generalized Bell inequality with an arbitrary number of settings. Specific local realistic models, which predict n-setting models, must satisfy it. In the next section, we shall show that if one replaces E_{LR} by E one may have a violation of the inequality (32). One has

$$\begin{aligned}(E, E) &= \sum_{l_1=1}^{n}\sum_{l_2=1}^{n}\cdots\sum_{l_N=1}^{n}\left(\sum_{i_1,i_2,\ldots,i_N=1,2} T_{i_1 i_2 \ldots i_N} c_1^{i_1} c_2^{i_2} \cdots c_N^{i_N}\right)^2 \\ &= \left(\frac{n}{2}\right)^N \sum_{i_1,i_2,\ldots,i_N=1,2} T^2_{i_1 i_2 \ldots i_N}.\end{aligned} \tag{33}$$

Here, we have used the fact that $\sum_{l_j=1}^{n} c_j^{\alpha} c_j^{\beta} = \frac{n}{2}\delta_{\alpha,\beta}$, because $c_j^1 = \cos\alpha_j^{l_j}$ and $c_j^2 = \sin\alpha_j^{l_j}$.

The structure of the condition (32) and the value (33) suggests that the value of (33) does not have to be smaller than (32). That is there may be such correlation functions E, which have the property that for any E_{LR} (n-setting model) one has $(E_{LR}, E) < (E, E)$, which implies impossibility of modeling E by "n-setting" local realistic correlation function E_{LR} with respect to the n measurement directions.

3 Further Detailed Classification of Local Realistic Theories

We present here an important example of a violation of the condition (32). The property of two-setting model is different from the property of n-setting model. And, the property of M'-setting model is different from the property of M-setting model ($M' < M$).

Imagine N observers who can choose between two orthogonal directions of spin measurement, $\vec{x}_j^{(1)}$ and $\vec{x}_j^{(2)}$ for the jth one. Let us assume that the source of N entangled spin-carrying particles emits them in a state, which can be described as a generalized Werner state, namely $V|\psi_{GHZ}\rangle\langle\psi_{GHZ}| + (1-V)\rho_{noise}$, where $|\psi_{GHZ}\rangle = 1/\sqrt{2}(|+\rangle_1\cdots|+\rangle_N + |-\rangle_1\cdots|-\rangle_N)$ is the Greenberger, Horne, and Zeilinger (GHZ) state [23] and $\rho_{noise} = \frac{1}{2^N}\mathbb{1}$ is the random noise admixture. The value of V can be interpreted as the reduction factor of the interferometric contrast observed in the multi-particle correlation experiment. The states $|\pm\rangle_j$ are the eigenstates of the σ_z^j observable. One can easily show that if the observers limit their settings to $\vec{x}_j^{(1)} = \hat{x}_j$ and $\vec{x}_j^{(2)} = \hat{y}_j$ there are 2^{N-1} components of $\hat{T}$ of the value $\pm V$. These are $T_{11\ldots1}$ and all components that except from indices 1 have an even number of indices 2. Other x–y components vanish.

It is easy to see that $T_{max} = V$ and $\sum_{i_1,i_2,\ldots,i_N=1,2} T^2_{i_1 i_2 \ldots i_N} = V^2 2^{N-1}$. Then, we have

$$(E_{LR}, E) \leq \left(\frac{1}{\sin(\frac{\pi}{2n})}\right)^N V$$

and

$$(E, E) = \left(\frac{n}{2}\right)^N V^2 2^{N-1} = n^N V^2 \frac{1}{2}.$$

For $N \geq 6$ and V given by

$$2\left(\frac{1}{n\sin\left(\frac{\pi}{2n}\right)}\right)^N < V \leq \frac{1}{\sqrt{2^{N-1}}} \tag{34}$$

despite the fact that there exist "two-setting" local realistic models for n measurement directions in consideration

$$(A_1, A_2, \ldots, A_n) \equiv \left(0, \frac{\pi}{n}, \ldots, \frac{(n-1)\pi}{n}\right) \tag{35}$$

these models cannot construct "n-setting" local realistic models $(A_1, A_2, \ldots, A_n)$. Namely, even though there exist two-setting models for a set of measurement directions $(A_1, A_2), (A_2, A_3), \ldots, (A_n, A_1)$, these models cannot construct any n-setting models for $(A_1, A_2, \ldots, A_n)$.

As it was shown in [22] if the correlation tensor satisfies the following condition

$$\sum_{i_1, i_2, \ldots, i_N = 1,2} T^2_{i_1 i_2 \ldots i_N} \leq 1 \tag{36}$$

then there always exists "two-setting" local realistic model for the set of correlation function values for all directions lying in a plane. For our example the condition (36) is met whenever $V \leq \frac{1}{\sqrt{2^{N-1}}}$. Nevertheless such models cannot construct "n-setting" local realistic models for $V > 2(\frac{1}{n\sin(\frac{\pi}{2n})})^N$. Thus the situation is such for $V \leq \frac{1}{\sqrt{2^{N-1}}}$ for all two settings per observer experiments one can construct "two-setting" local realistic model for the values of the correlation function for the settings chosen in the experiment. One wants to construct "n-setting" local realistic model for n measurement directions $(A_1, A_2, \ldots, A_n)$ by using "two-setting" local realistic models, $(A_1, A_2), (A_2, A_3), \ldots, (A_n, A_1)$. But these n number of "two-setting" models must be consistent with each other, if we want to construct truly "n-setting" local realistic models beyond the 2^N settings to which each of them pertains. Our result clearly indicates that this is impossible for $V > 2(\frac{1}{n\sin(\frac{\pi}{2n})})^N$. These "two-setting" local realistic models, $(A_1, A_2), (A_2, A_3), \ldots, (A_n, A_1)$ must contradict each other. Rather they are therefore invalidated. In other words the explicit two-setting models, given in [22] work only for the specific set of settings in the given experiment, but cannot construct a local realistic model for the values of a correlation function, given in a n-setting Bell experiment (n-setting model), even though there exist two-setting models for the n measurement directions chosen in the given n-setting experiment. Therefore, the property of two-setting model is different from the property of n-setting model.

One can see that M'-setting model does not have the property which M-setting model has $(M' < M)$ when

$$2\left(\frac{1}{M\sin\left(\frac{\pi}{2M}\right)}\right)^N < V \leq 2\left(\frac{1}{M'\sin\left(\frac{\pi}{2M'}\right)}\right)^N. \tag{37}$$

This condition implies that a violation of the assumption of the existence of M-setting model allows the assumption of the existence of M'-setting model. Thus, the property of M'-setting model is different from the property of M-setting model.

For the continuous range of settings $(n \to \infty)$ one recovers the result of [17]:

$$2\left(\frac{2}{\pi}\right)^N < V \leq \frac{1}{\sqrt{2^{N-1}}}. \tag{38}$$

For three settings per site ($n = 3$) the result of [21] is obtained:

$$2\left(\frac{2}{3}\right)^N < V \leq \frac{1}{\sqrt{2^{N-1}}}. \tag{39}$$

Please note that all information needed to get this conclusion can be obtained in two-orthogonal-setting-per-observer experiments, that is with the information needed in the case of “standard” two settings Bell inequalities [22, 24–28]. To get both the value of (33) and of T_{max} it is enough to measure all values of $T_{i_1 i_2 \ldots i_N}$, $i_1, i_2, \ldots, i_N = 1, 2$.

4 Conclusions and Discussions

In conclusion, we have derived a generalized Bell inequality for N qubits which involves an arbitrary number of settings for each of the local measuring apparatuses. The inequality has formed a necessary condition for the existence of a local realistic model for the values of a correlation function, given in a n-setting Bell experiment. And we have shown that a local realistic model for the values of a correlation function, given in a two-setting Bell experiment, cannot construct a local realistic model for the values of a correlation function, given in a n-setting Bell experiment, even though there exist two-setting models for the n measurement directions chosen in the given n-setting experiment. Therefore, the property of two-setting model has been different from the property of n-setting model.

Our result have provided classification of local realistic theories in further detail (more than the result presented in Ref. [21]). We can have seen that there are many types of local realistic theories along with measurement settings. First, there has been two-setting model. It has been explicitly constructed by standard two-setting Bell inequalities [22]. However, this model has been disproved by several generalized Bell inequalities. The patterns of the disqualification have been different form each other along with measurement settings. Therefore, one furthermore has had more different types of models. These have been n-setting model ($3 \leq n$), plane-infinite-model, and sphere-infinite-model, as we have mentioned above.

Locality is a consequence of the general symmetries of the Poincaré group of the Special Relativity Theory. However it is a direct consequence of the Lorentz transformations (boosts), as they define the light-cone. As our discussion shows a subgroup of the Poincaré group, symmetries of discrete rotations of the Cartesian coordinates, introduces an additional classification of local realistic theories.

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